ON THE APPROXIMATE SOLUTIONS IN INTEGERS OF A SET OF LINEAR EQUATIONS

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1. It has been proved that a set of $n+1$ integers[1] $x_1, x_2, \ldots, x_n, w$ exist which satisfy approximately the n equations[2]

$$x_1 - \alpha_1 w - \beta_1 = 0, \ldots, x_n - \alpha_n w - \beta_n = 0, \qquad (1)$$

where $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ are $2n$ given real numbers, the first n of which are subject to a condition (C) stated below. The degree of approximation is as follows: having specified a positive number ϵ as small as we please, then the numerical values of the left-hand members of the above equations (i.e., the "errors") can simultaneously be made smaller than ϵ. The condition (C) imposed upon $\alpha_1, \alpha_2, \ldots. \alpha_n$ is this: they are to be irrational numbers of such a nature that a linear expression of the form $a_0 + a_1\alpha_1 + \ldots + a_n\alpha_n$, with integral coefficients $a_0, a_1, \ldots, a_n$, can vanish exactly *only* when these coefficients are all $= 0$.

For a given approximate solution of the equations (1) the errors shall be designated, respectively, by $\epsilon_1, \ldots, \epsilon_n$.

2. In the two special cases: (I) $n = 1$, or (II) $\beta_1 = \beta_2 = \ldots = \beta_n = 0$, we can make additional demands upon the errors, namely (and this is of importance in applications) we can cause them to be smaller than certain corresponding functions of w. Thus, in the case (I), we can demand that $\epsilon_1 < \epsilon$ and at the same time that $\epsilon_1 < 1/(4|w|)$; in case (II) we can demand that $\epsilon_1, \ldots, \epsilon_n$ be each $< \epsilon$ and at the same time $< k/(|w|)^m$, where $k = n/(n+1)$, and $m = -(n-1)/n$; furthermore, the condition (C) is not necessary in case (II).[3] These results may be stated in another way: in case (I), α being irrational, an infinite number of sets of integers $(x', w'), (x'', w''), \ldots.$ exist which, when substituted in turn in $x - \alpha w - \beta = 0$, will produce errors $\epsilon', \epsilon'', \ldots.$ of such magnitudes that $\epsilon' < 1/(4|w'|)$, $\epsilon'' < 1/(4|w''|)$, etc. Similarly, in case (II) we have an infinite number of solutions satisfying corresponding inequalities.

3. In the general case the errors cannot be made to satisfy such extra demands. Stated more precisely: *no matter what function $f(w)$ be assumed, if only it be subjected to the conditions* (D) *below, we can always construct a set of equations* (1) *with attendant condition* (C), *which do not admit an infinite number of approximate solutions in integers* $(x_1, x_2, \ldots, x_n), \ldots.$, *if we demand the following degree of accuracy for each solution:* $\epsilon_1 < f(|w|), \ldots, \epsilon_n < f(|w|)$. *In fact, we may even substitute the weaker demand.*

$$\epsilon_2\epsilon_3\ldots\epsilon_n + \epsilon_1\epsilon_3\ldots\epsilon_n + \ldots + \epsilon_1\epsilon_2\ldots\epsilon_{n-1} < f(\,|\,w\,|\,). \qquad (2)$$

The conditions (D) referred to are: for every positive integer w, $f(w)$ is a positive number which decreases when w increases, and approaches 0 as a limit when w tends to ∞.

4. We proceed to give an outline of the proof, limiting ourselves to the case $n = 3$, which is sufficiently illustrative. The function $f(w)$ being given subject to (D), we shall take as our equations the following set (under an obvious change of notation):

$$x - \alpha w - r\alpha = 0,\ y - \beta w - s\beta = 0,\ z - \gamma w - t\gamma = 0. \qquad (3)$$

Here r, s, t represent three arbitrary, but different integers, while α, β, γ are defined as the limits approached when j tends to ∞ by three series of generalized (Jacobian) continued fractions x_j/w_j, y_j/w_j, z_j/w_j; $j = 1, 2, \ldots$. We take at the outset any four sets of non-negative integers $(x_1, \ldots, w_1)$, $\ldots$, $(x_4, \ldots, w_4)$, such that the determinant $(x_1y_2z_3w_4) = 1$; succeeding sets $(x_j, \ldots, w_j;\ j > 4)$ are constructed from previous sets by the rule: $x_j = s_{j-1}x_{j-1} + x_{j-4}$, $y_j = s_{j-1}y_{j-1} + y_{j-4}$, etc., involving a series of positive integers $s_4, s_5, s_6, \ldots$, which, in turn, are expressed in terms of another series designated by [3], [4], [5], $\ldots$; namely, we put $s_j = 2^{[j]-[j-1]}$. The members of the new series are any set of positive integers satisfying the following conditions:

$[3] = 1$; when $j > 3$ take $[j] \geqq 3[j-1]$, $f(2^{[j]-[j-1]}) < 2^{-6[j-1]}$.

5. For convenience in the further development we make use of the letters M, m, μ to represent certain non-negative coefficients (constants or variables), the actual values of which are not required: M for a number having a lower bound > 0, m for one having an upper bound, and μ for one having both a lower bound > 0 and an upper bound.

We note the following preliminary relations:

$x_j = \mu.2^{[j-1]}$, $y_j = \mu.2^{[j-1]}$, etc.;

$\xi_j \equiv x_j - \alpha w_j = \pm m/s_j$, $\eta_j \equiv y_j - \beta w_j = \pm m/s_j$, $\zeta_j = z_j - \gamma w_j = \pm m/s_j$; $x_iw_j - x_jw_i = (x_iw_j) = \pm m.2^{[j-1]-[i]+[i-1]}$ when $j > i$, $y_iw_j - y_jw_i = (y_iw_j) =$ etc., etc. Moreover, (x_iw_j), etc., do not vanish when j is taken sufficiently high.

(The condition (C), §1, is satisfied by α, β, γ. For, an equation $a_0 + a_1\alpha + a_2\beta + a_3\gamma = 0$ would imply $a_0w_j + a_1x_j + a_2y_j + a_3z_j = 0$ for every subscript j above a certain number. But this requires $a_0 = a_1 = a_2 = a_3 = 0$, since the determinant $(x_jy_{j+1}z_{j+2}w_{j+3}) = \pm 1$.)

6. If x, y, z, w represent any four integers, then at least two of the numbers

$$A = xw_j - x_j(w+r),\quad B = yw_j - y_j(w+s),\quad C = zw_j - z_j(w+t)$$

are of magnitude $\pm M.2^{[j-1]-2[j-2]}$, when j exceeds a certain number. For selecting any pair, say A, B, we have

$$(Aq - Bp)\,w_{j-1} + (r - s)pq \equiv 0 \qquad (\text{mod. } w_j),$$

where $p=(x_j w_{j-1})$, $q=(y_j w_{j-1})$. But this congruence is found to imply $|Aq - Bp|\, w_{j-1} > wj$ for sufficiently high values of j. Hence, either $|Aq|$ or $|Bp|$ or both are $> w_j/w_{j-1}$.

7. Reverting now to the equations (3), we take any approximate solution in integers, say x, y, z, w. Assuming for our purpose that $|w| \geqq 2^{[4]-[3]}$, we determine the positive integer j such that

$$2^{[j-1]-[j-2]} \leqq |w| < 2^{[j]-[j-1]}.$$

We have

$$w_j\epsilon_1 = |A+(w+r)\xi_j|, \quad w_j\epsilon_2 = |B+(w+s)\eta_j|, \quad w_j\epsilon_3 = |C+(w+t)\zeta_j|,$$

and it follows from the results above that at least one of the three products $\epsilon_1\epsilon_2$, $\epsilon_2\epsilon_3$, $\epsilon_3\epsilon_1$ is of magnitude $M.2^{-4[j-2]}$ for sufficiently large values of j. But, under the same condition,

$$M.2^{-4[j-2]} > 2^{-6[j-2]} > f(2^{[j-1]-[j-2]}) \geqq f(|w|).$$

Hence, when j exceeds a certain number, the condition (2) is not fulfilled. That is, the values of w satisfying this condition are limited.

[1] Throughout this paper the term "integer" means "a positive or negative integer or zero," when it is not specially defined.

[2] Kronecker, L., *Monatsberichte Kgl. Preuss. Aka. Wiss.*, 1884 (1179ff., 1271ff.); *Werke*, **3**$^{\mathrm{I}}$, Leipzig, 1899 (49–109).

[3] See Dickson, L. E., *History of the Theory of Numbers*, **2**; *Carnegie Institution, Wash.*, 1920 (93–99), for references.

A PHYSICAL BASIS FOR EPIDEMIOLOGY

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In the autumn of 1918 there swept through one of the mouse breeding rooms of the Rockefeller Institute a destructive epidemic of mouse typhoid—an infection of mice with a bacillus of the enteritidis group of organisms to which the name of *Bacillus typhi murium* has been given. The history of the epidemic is instructive: the original mice of the population, numbering about 3000 at the time of the epidemic, came from a breeder in Massachusetts and had been purchased some time before and moved *en masse* to the Rockefeller Institute. In the meantime many new mice had been born of this stock, and many of the original mice had died or been employed for experiment, so that only a small residue of the original population remained. In other words, the epidemic of mouse typhoid arose among chiefly a new stock of mice, the offspring of an old stock believed on good grounds to have passed through previous outbreaks of the disease.

There is still another reason for supposing that the epidemic arose from within and was not imported from without this stock. Besides the breed-